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J. VAN DE LUNE & H.J.J. TE RIELE

A NOTE ON THE PARTIAL SUMS OF $\zeta(s)$, III

2e boerhaavestraat 49 amsterdam

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A note on the partial sums of $\zeta(s)$, III

by

J. van de Lune & H.J.J. te Riele

ABSTRACT

This note is a continuation of the Mathematical Centre Reports ZW 53/75 and ZW 58/75.

For $N \geq 2$ let $\zeta_N(s) = \sum_{n=1}^{N} n^{-s}$, $(s\in\mathbb{C})$. It is shown that for $N = 2(1)6$ and $N = 8(1)10$ the functions ζ_N have no zeros in the half-plane $\sigma = \mathrm{Re}(s) > 1$.

In addition it is shown that all zeros of ζ_N lie in the strip $1-N \log 2 < \sigma < 1.72865$.

With respect to the orders of the zeros of ζ_N it is shown that for every N there exists a number $\omega(N)$ such that the orders of the zeros of ζ_N do not exceed $\omega(N)$. For $N = 2(1)4$ all zeros of ζ_N are shown to be simple.

Finally an improvement is given of a theorem of SPIRA on the location of the lowest zero of ζ_N for large N.

KEY WORDS & PHRASES: *Partial sums (sections) of the zeta-function, zeros.*

0. INTRODUCTION.

This note is a continuation of the Mathematical Centre Reports ZW 53/75 and ZW 58/75.

In section 1 we present a device by means of which one may easily show that for $N = 2(1)6$ and $N = 8(1)10$ the functions ζ_N defined by

$$\zeta_N(s) = \sum_{n=1}^{N} n^{-s}, \qquad (s\in\mathbb{C};\ N=2,3,4,\ldots) \tag{0.1}$$

have no zeros in the half-plane $\sigma = \mathrm{Re}(s)>1$. (Compare the methods of JESSEN and SPIRA described in TURAN [8], SPIRA [6] and SPIRA [7]).

Section 2 is concerned with the width of the vertical strip in $\mathbb{C}$ containing all zeros of $\zeta_N(s)$.

In SPIRA [6] it was shown that for $N \geq 2$ all zeros of $\zeta_N(s)$ satisfy

$$\sigma > 1 - N. \tag{0.2}$$

It will be shown here that this result may be improved by utilizing some new inequalities for sums of powers of positive integers (also see [4]). As a result we have that (0.2) may be replaced by

$$\sigma > 1 - \frac{\log 2}{\log(1+\frac{1}{N-\frac{1}{2}})}. \tag{0.3.1}$$

Also it will be shown that (roughly speaking) the south-west corner of the halfstrip

$$\sigma > 1 - \frac{\log 2}{\log(1+\frac{1}{N-\frac{1}{2}})}, \qquad \mathrm{Im}(s) = t \geq 0 \tag{0.3.2}$$

is a zero-free region for $\zeta_N(s)$.

For a more precise formulation see theorem 2.2.

Section 3 deals with the multiplicity of the zeros of $\zeta_N(s)$. It will be shown that for every $N \geq 2$ there exists a number $\omega(N)$ such that the multiplicity of any zero of $\zeta_N(s)$ does not exceed $\omega(N)$.

In particular it will be proved that for $N = 2(1)4$ we may take $\omega(N) = 1$ so that for these values of N all zeros of $\zeta_N(s)$ are simple.

It is *conjectured* that for all $N \geq 2$ all zeros of $\zeta_N(s)$ are simple.

Section 4 deals with the location of the "lowest" zero of $\zeta_N(s)$.

In SPIRA [5] it was shown that if N is large enough then the lowest zero of $\zeta_N(s)$ is simple and lies either in the rectangle

$$(0.4)\qquad \begin{cases} 1 - \dfrac{c_1}{\log^3 N} \leq \sigma \leq 1 + \dfrac{c_2}{\log^3 N}, \\[2ex] \dfrac{2\pi}{\log N} - \dfrac{c_3}{\log^2 N} \leq t \leq \dfrac{2\pi}{\log N} \end{cases}$$

or in the triangle

$$(0.5)\qquad \begin{cases} 1 \leq \sigma \leq 1 + \dfrac{c_4}{\log^3 N}, \\[2ex] \dfrac{2\pi}{\log N} \leq t \leq \dfrac{2\pi}{\log N} + \dfrac{\sigma - 1}{\log^2 N}. \end{cases}$$

For the positive constants c_1, c_2, c_3 and c_4, see SPIRA [5].

We will show that if N is large enough then the region (0.5) does *not* contain any zero of $\zeta_N(s)$.

For more results on this subject see LEVINSON [1].

1. In [2] and [3] it was shown that for $N = 2(1)6$ and $N = 8(1)10$ there exist positive constants $p(N)$ such that

$$(1.1)\qquad R_N(t) \overset{\text{def}}{=} \operatorname{Re}\zeta_N(1+it) \geq p(N), \qquad (t \in \mathbb{R}).$$

Defining

$$(1.2)\qquad R_N(\sigma,t) = \operatorname{Re}\zeta_N(\sigma+it) = \sum_{n=1}^{N} \frac{\cos(t \log n)}{n^\sigma}$$

we clearly have

$$(1.3) \qquad R_N(\sigma,0) = \sum_{n=1}^{N} n^{-\sigma} > 0, \qquad (\sigma\in\mathbb{R})$$

and

$$(1.4) \qquad R_N(\sigma,t) \geq 1 - \sum_{n=2}^{N} n^{-\sigma} > 2 - \zeta(\sigma) \geq$$

$$\geq 2 - \zeta(2) = 2 - \frac{\pi^2}{6} > 0, \qquad (\sigma\geq 2;\ t\in\mathbb{R}) .$$

Next observe that $R_N(t)$ is an almost-periodic function so that for every $\varepsilon > 0$ there exists an increasing positive sequence $\left\{T_k\right\}_{k=1}^{\infty}$ such that $T_k \to \infty$ as $k \to \infty$ and

$$(1.5) \qquad R_N(T_k) > R_N(0) - \varepsilon.$$

If we choose ε small enough it follows that in the sum

$$(1.6) \qquad R_N(T_k) = \sum_{n=1}^{N} \frac{1}{n} \cos(T_k \log n)$$

all cosines must be close to 1 (and hence positive) so that also

$$(1.7) \qquad R_N(\sigma,T_k) = \sum_{k=1}^{N} \frac{1}{n^{\sigma}} \cos(T_k \log n) > 0, \qquad (\sigma\in\mathbb{R}).$$

Finally observe that $R_N(\sigma,t)$ is a harmonic function so that by the minimum principle for harmonic functions we easily obtain the following

THEOREM 1.1. *For* $N = 2(1)6$ *and* $N = 8(1)10$ *we have*

$$(1.8) \qquad R_N(\sigma,t) > 0, \qquad (\sigma\geq 1,\ t\geq 0) .$$

As a simple consequence we have

THEOREM 1.2. *For* $N = 2(1)6$ *and* $N = 8(1)10$ *we have*

(1.9) $\zeta_N(s) \neq 0 \quad \textit{for} \quad \sigma \geq 1.$

Similarly one may prove

<u>THEOREM 1.3.</u> *If* $R_N(\sigma_0, t) > 0$, $(t \in \mathbb{R})$ *then* $\zeta_N(s) \neq 0$ *for* $\sigma \geq \sigma_0$.

The last theorem can easily be extended to functions $D_N(s) = \sum_{n=1}^{N} a_n n^{-s}$ *with* $a_1 > 0$ *and* $a_n \geq 0$ *for* $1 < n \leq N$.

<u>REMARK.</u> In order to prove theorem 1.1 it suffices to show that

$$R_N(\sigma, t) > 0, \qquad (1 \leq \sigma \leq 2,\ t \geq 0)$$

for all N stated in the theorem. Indeed, by (1.4) we already know that

$$R_N(\sigma, t) > 0, \qquad (\sigma \geq 2;\ t \in \mathbb{R}).$$

Observing that

$$\frac{\partial R_N(\sigma,t)}{\partial \sigma} = - \sum_{n=2}^{N} \frac{\log n}{n^\sigma} \cos(t \log n) \tag{1.10}$$

we have

$$\left| \frac{\partial R_N(\sigma,t)}{\partial \sigma} \right| \leq \sum_{n=2}^{N} \frac{\log n}{n^\sigma} \leq \sum_{n=2}^{N} \frac{\log n}{n} \tag{1.11}$$

for $\sigma \geq 1$ and all $t \in \mathbb{R}$.
Hence, by the maximal slope principle, if

$$\sum_{n=2}^{N} \frac{\log n}{n} < \sum_{n=1}^{N} \frac{1}{n} - \varepsilon = R_N(0) - \varepsilon, \tag{1.12}$$

then

$$R_N(\sigma, T_k) \geq R_N(T_k) - (\sigma - 1) \sum_{n=2}^{N} \frac{\log n}{n} > \tag{1.13}$$

$$> R_N(0) - \varepsilon - \sum_{n=2}^{N} \frac{\log n}{n}, \qquad (1 \leq \sigma \leq 2).$$

Since $\varepsilon > 0$ may be chosen as small as we please it suffices to show that

$$L_N \overset{\text{def}}{=} \sum_{n=2}^{N} \frac{\log n}{n} < \sum_{n=1}^{N} \frac{1}{n} = R_N(0). \tag{1.14}$$

It is easily verified (see table 1) that (1.14) is true indeed for $N = 2(1)10$ so that we have obtained an alternative proof of theorem 1.1.

TABLE 1

N	L_N	$R_N(0)$
2	.3466	1.5000
3	.7128	1.8333
4	1.0594	2.0833
5	1.3812	2.2833
6	1.6799	2.4500
7	1.9579	2.5929
8	2.2178	2.7179
9	2.4619	2.8290
10	2.6922	2.9290
11	2.9102	3.0199
12	3.1172	3.1032
13	3.3145	3.1801
14	3.5031	3.2516
15	3.6836	3.3182

2. <u>THEOREM 2.1.</u> *If* $\zeta_N(s) = 0$ *then*

$$\sigma = \mathrm{Re}(s) > 1 - \frac{\log 2}{\log(1+\frac{2}{2N-1})}. \tag{2.1}$$

In order to prove this we first establish the following

<u>Lemma 2.1.</u> *For all real* $p \geq 1$ *we have*

$$\sigma_n(p) \overset{\text{def}}{=} \sum_{k=1}^{n} k^p < \frac{n^p(2n+1)^{p+1}}{(2n+1)^{p+1}-(2n-1)^{p+1}}. \tag{2.2}$$

<u>Proof of the lemma.</u>

It is easily verified that (2.2) is true for $n = 1$ and all $p \geq 1$. Now assume that the lemma is true for $n = 1,\ldots,N$ and all $p \geq 1$. Then we have

$$\sigma_{N+1}(p) = (N+1)^p + \sigma_N(p) < (N+1)^p + \frac{N^p(2N+1)^{p+1}}{(2N+1)^{p+1}-(2N-1)^{p+1}}, \tag{2.3}$$

so that it suffices to show that

$$(N+1)^p + \frac{N^p(2N+1)^{p+1}}{(2N+1)^{p+1}-(2N-1)^{p+1}} \leq \frac{(N+1)^p(2N+3)^{p+1}}{(2N+3)^{p+1}-(2N+1)^{p+1}} \tag{2.4}$$

for all $N \in \mathbb{N}$ and all $p \geq 1$.
Putting $\frac{1}{N} = x$ we arrive at the equivalent inequality

$$(1+x)^p + \frac{(2+x)^{p+1}}{(2+x)^{p+1}-(2-x)^{p+1}} \leq \frac{(1+x)^p(2+3x)^{p+1}}{(2+3x)^{p+1}-(2+x)^{p+1}}. \tag{2.5}$$

Replace x by $2x$ (so that from now on $0 < x \leq \frac{1}{2}$) in order to arrive at the equivalent inequality

$$(1+2x)^p + \frac{(1+x)^{p+1}}{(1+x)^{p+1}-(1-x)^{p+1}} \leq \frac{(1+2x)^p(1+3x)^{p+1}}{(1+3x)^{p+1}-(1+x)^{p+1}}. \tag{2.6}$$

After crossmultiplication and some simplification it turns out that we may just as well prove that

$$(2.7)\qquad (1+2x)^p\left\{(1+x)^{p+1}-(1-x)^{p+1}\right\}\geq(1+3x)^{p+1}-(1+x)^{p+1}$$

which is equivalent to

$$(2.8)\qquad \frac{(1+x)^{p+1}-(1-x)^{p+1}}{x}\geq\frac{(1+\frac{x}{1+2x})^{p+1}-(1-\frac{x}{1+2x})^{p+1}}{\frac{x}{1+2x}}.$$

Since $x > \frac{x}{1+2x}$ for $x > 0$ it follows that the proof is complete as soon as we can show that the function

$$(2.9)\qquad \varphi(x)\overset{\text{def}}{=}\frac{(1+x)^{p+1}-(1-x)^{p+1}}{x}$$

is increasing on the interval $0 < x \leq \frac{1}{2}$.
Observe that for $x > 0$ we have

$$(2.10)\qquad \varphi'(x)=\frac{x\left\{(p+1)(1+x)^p+(p+1)(1-x)^p\right\}-(1+x)^{p+1}+(1-x)^{p+1}}{x^2}$$

so that it suffices to show that

$$(2,11)\qquad (px+x-1-x)(1+x)^p+(px+x+1-x)(1-x)^p\geq 0$$

or, equivalently, that

$$(2.12)\qquad f(x)\overset{\text{def}}{=}(px-1)(1+x)^p+(px+1)(1-x)^p\geq 0.$$

Since $f(0) = 0$, it suffices to show that for $x > 0$

$$(2.13)\qquad f'(x)=p(1+x)^p+(px-1)p(1+x)^{p-1}+p(1-x)^p-(px+1)p(1-x)^{p-1}\geq 0$$

or, equivalently, that

$$(2.14) \qquad (p+px+p^2x-p)(1+x)^{p-1} + (p-px-p^2x-p)(1-x)^{p-1} =$$

$$= (p+p^2)x\left\{(1+x)^{p-1}-(1-x)^{p-1}\right\} \geq 0.$$

Since (2.14) is clearly true for $p \geq 1$ and $0 < x < 1$ the proof of the lemma is complete.

Proof of theorem 2.1.

The case $N = 2$ being trivial we assume $N \geq 3$.

Put

$$(2.15) \qquad p = \frac{\log 2}{\log(1+\frac{2}{2N-1})} - 1$$

and assume the theorem to be false. Then there exists an $N \geq 3$ and an $s \in \mathbb{C}$ such that

$$(2.16) \qquad \sigma = \mathrm{Re}(s) \leq -p$$

and

$$(2.17) \qquad \zeta_N(s) = \sum_{n=1}^{N} n^{-s} = 0,$$

so that

$$(2.18) \qquad \sum_{n=1}^{N-1} n^{-s} = -N^{-s}.$$

It follows that

$$(2.19) \qquad -1 = \sum_{n=1}^{N-1} \left(\frac{N}{n}\right)^s,$$

so that

$$(2.20) \qquad 1 \le \sum_{n=1}^{N-1} \left(\frac{N}{n}\right)^{\sigma}.$$

In (2.20) we have $1 \le n \le N - 1$ so that $\frac{N}{n} > 1$. Since $\sigma \le -p$ we have

$$(2.21) \qquad 1 \le \sum_{n=1}^{N-1} \left(\frac{N}{n}\right)^{-p} = \sum_{n=1}^{N-1} \left(\frac{n}{N}\right)^{p}$$

or, equivalently,

$$(2.22) \qquad 2\, N^{p} \le \sigma_N(p).$$

It is easily verified that $p > 1$ for $N \ge 3$. Hence, we may apply lemma 2.1 to (2.22) in order to obtain

$$(2.23) \qquad 2\, N^{p} < \frac{N^{p}(2N+1)^{p+1}}{(2N+1)^{p+1} - (2N-1)^{p+1}},$$

from which it is easily seen that

$$(2.24) \qquad p + 1 < \frac{\log 2}{\log(1 + \frac{2}{2N-1})},$$

contradicting the definition of p and hence proving the theorem.

<u>REMARK.</u> In [4] it was shown by very simple means that for $p > 0$ one has

$$(2.25) \qquad \sigma_N(p) < \frac{N^{p}(N+1)^{p+1}}{(N+1)^{p+1} - N^{p+1}}.$$

Similarly as above, from this inequality one may derive that if $\zeta_N(s) = 0$ then

$$(2.26) \qquad \sigma > 1 - \frac{\log 2}{\log(1 + \frac{1}{N})},$$

a result just a little less sharp than (2.1).

We conclude this section by proving

THEOREM 2.2. *$\zeta_N(s)$ has no zeros in the domain G_N described by*

$$(2.27)\qquad \begin{cases} s = \sigma + a\sigma i; & a, \sigma \in \mathbb{R} \\ \sigma \le -1 \\ |s-1| \le \dfrac{2(N-1)}{1+\sqrt{1+a^2}} \end{cases}$$

PROOF. If $\zeta_N(s) = 0$ then we have

$$(2.28)\qquad 0 = \zeta_N(s) = \sum_{n=1}^{N} n^{-s} = \int_{1-}^{N+} x^{-s}\, d\,[x] =$$

$$= \int_{1}^{N} x^{-s}\, dx - \int_{1-}^{N+} x^{-s}\, d\,\varphi_1(x) =$$

(where $\varphi_1(x) = x - [x] - \frac{1}{2}$)

$$= \frac{N^{-s+1}-1}{-s+1} + \frac{1}{2}(N^{-s}+1) - s\int_{1}^{N} \frac{\varphi_1(x)}{x^{s+1}}\, dx$$

so that

$$(2.29)\qquad \frac{1-N^{s-1}}{s-1} = \frac{1}{2N} + \frac{1}{2}N^{s-1} - s\,N^{s-1}\int_{1}^{N} \frac{\varphi_1(x)}{x^{s+1}}\, dx.$$

It follows that we must have

$$(2.30)\qquad \frac{1-N^{\sigma-1}}{|s-1|} \le \frac{1}{2N} + \frac{1}{2}N^{\sigma-1} + |s|\,N^{\sigma-1}\int_{1}^{N} \frac{|\varphi_1(x)|}{x^{\sigma+1}}\, dx$$

$$\le \frac{1}{2N} + \frac{1}{2}N^{\sigma-1} + |s|\,N^{\sigma-1}\,\frac{1}{2}\,\frac{N^{-\sigma}-1}{-\sigma}$$

$$= \frac{1}{2N} + \frac{1}{2}N^{\sigma-1} + \frac{|s|}{-2\sigma}\left(\frac{1}{N} - N^{\sigma-1}\right).$$

Now let $s = \sigma + a\sigma i$ with $\sigma \leq -1$ and $a \in \mathbb{R}$. Then we obtain

$$(2.31)\qquad \frac{1-N^{-2}}{|s-1|} > \frac{1}{2N} + \frac{1}{2N^2} + \frac{|s|}{2N|\sigma|}$$

so that

$$(2.32)\qquad \frac{N^2-1}{|s-1|} > \frac{N+1}{2} + \frac{N}{2}\sqrt{1+a^2} < \frac{N+1}{2}(1+\sqrt{1+a^2}).$$

Hence

$$(2.33)\qquad |s-1| > \frac{2(N-1)}{1+\sqrt{1+a^2}},$$

proving the theorem.

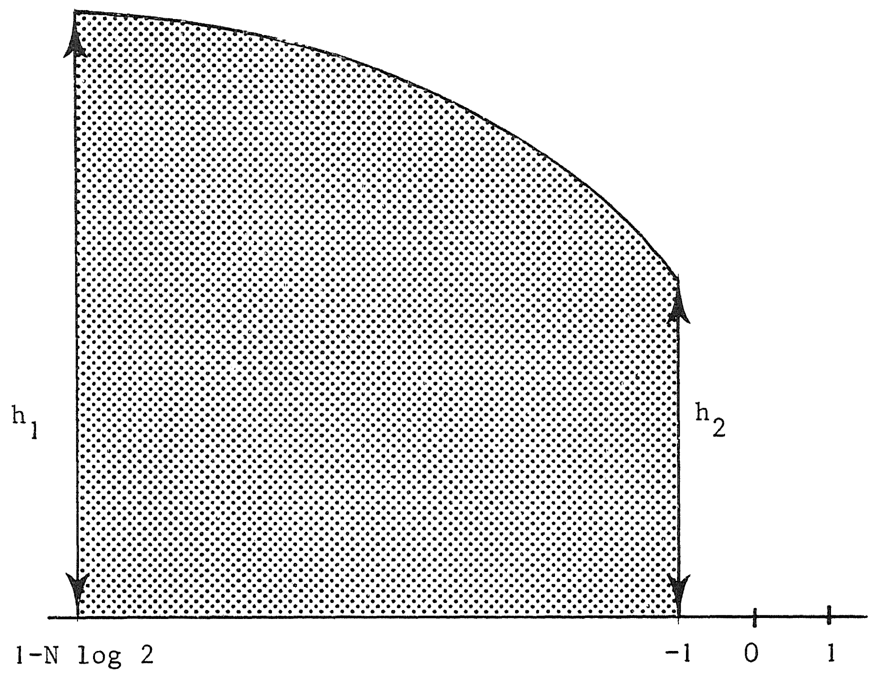
<u>REMARK.</u> Since

$$(2.34)\qquad 1 - N\log 2 < 1 - \frac{\log 2}{\log(1+\frac{1}{N-\frac{1}{2}})}, \qquad (\forall N \in \mathbb{N})$$

we derive from theorems 2.1 and 2.2 that the intersection of the domain G_N in theorem 2.2 and the half-strip

$$(2.35)\qquad \begin{cases} \sigma > 1 - N\log 2, \\ t \geq 0 \end{cases}$$

is a zero-free region for $\zeta_N(s)$. This region may be depicted as the shaded area in the figure below. One should compare these results with the empirical observations made by SPIRA [6; section 4].



From (2.7) it is easily seen that if N is large enough then

(2.36) $$h_1 > \tfrac{1}{2} N$$

and

(2.37) $$h_2 \sim \sqrt{2N} .$$

Finally we want to state that it is possible to improve theorem 2.2 in various ways. However, we will not pursue this subject here.

3. THEOREM 3.1. *For* N = 2(1)4 *all zeros of* $\zeta_N(s)$ *are simple.*

PROOF. The case N = 2 being trivial we first establish the case N = 3. If $\zeta_3(s)$ has a multiple zero then

(3.1) $$1 + \frac{1}{2^s} + \frac{1}{3^s} = 0$$

and

(3.2) $$\frac{\log 2}{2^s} + \frac{\log 3}{3^s} = 0.$$

From (3.2) it follows that

$$(3.3) \qquad \left(\frac{3}{2}\right)^s = -\frac{\log 3}{\log 2}$$

so that for some $k \in \mathbb{Z}$

$$(3.4) \qquad s = \frac{(2k+1)\pi i + \log \dfrac{\log 3}{\log 2}}{\log \frac{3}{2}} \, .$$

Since

$$(3.5) \qquad \frac{1}{3^s} = -\frac{\dfrac{\log 2}{\log 3}}{2^s}$$

it also follows that

$$(3.6) \qquad 1 + \frac{1}{2^s} - \frac{\dfrac{\log 2}{\log 3}}{2^s} = 0$$

so that

$$(3.7) \qquad 2^s = -\left(1 - \frac{\log 2}{\log 3}\right)$$

from which we obtain

$$(3.8) \qquad s = \frac{(2m+1)\pi i + \log\left(1 - \dfrac{\log 2}{\log 3}\right)}{\log 2} \, .$$

Since

$$(3.9) \qquad \frac{\log\left(1 - \dfrac{\log 2}{\log 3}\right)}{\log 2} < 0$$

and

$$(3.10) \qquad \frac{\log \dfrac{\log 3}{\log 2}}{\log \frac{3}{2}} > 0$$

we arrive at a contradiction.

We now prove the simplicity of the zeros of $\zeta_4(s)$. If $\zeta_4(s)$ has any multiple zero then

$$(3.11) \qquad 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} = 0,$$

and

$$\frac{\log 2}{2^s} + \frac{\log 3}{3^s} + \frac{\log 4}{4^s} = 0. \tag{3.12}$$

From the last equation it follows that

$$\frac{\frac{\log 2}{\log 3}}{2^s} + \frac{1}{3^s} + \frac{\frac{\log 4}{\log 3}}{4^s} = 0 \tag{3.13}$$

so that

$$1 + \frac{1}{2^s} - \frac{\frac{\log 2}{\log 3}}{2^s} - \frac{\frac{\log 4}{\log 3}}{4^s} + \frac{1}{4^s} = 0 \tag{3.14}$$

or

$$1 + (1 - \frac{\log 2}{\log 3})\frac{1}{2^s} + (1 - \frac{\log 4}{\log 3})(\frac{1}{2^s})^2 = 0. \tag{3.15}$$

The discriminant of the above quadratic equation in $\frac{1}{2^s}$ is

$$D = (1 - \frac{\log 2}{\log 3})^2 - 4(1 - \frac{\log 4}{\log 3}) > 0 \tag{3.16}$$

so that $\frac{1}{2^s}$ is real. It follows that also $\frac{1}{4^s}$ and hence also $\frac{1}{3^s}$ is real. Putting $2^s = x_2 \in \mathbb{R}$ and $3^s = x_3 \in \mathbb{R}$ we have

$$s = \frac{\log |x_2| + k\pi i}{\log 2} \text{ for some } k \in \mathbb{Z} \tag{3.17}$$

and

$$s = \frac{\log |x_3| + m\pi i}{\log 3} \text{ for some } m \in \mathbb{Z}. \tag{3.18}$$

It follows that for some $k, m \in \mathbb{Z}$

$$\frac{k}{\log 2} = \frac{m}{\log 3} \tag{3.19}$$

which is impossible unless $k = m = 0$. Hence s is real, which is clearly impossible. This contradiction proves that all zeros of $\zeta_4(s)$ are simple.

THEOREM 3.2. *For every* $N \geq 2$ *there is a number* $\omega(N)$ *such that the orders of the zeros of* $\zeta_N(s)$ *do not exceed* $\omega(N)$.

PROOF. From our previous work it is clear that we may take

$\omega(2) = \omega(3) = \omega(4) = 1$.

In general let $\zeta_N(s)$ have a zero of order $k + 1$.

Then

$$\begin{cases} \zeta_N(s) = 0, \\ \zeta_N'(s) = 0, \\ \dots\dots\dots, \\ \zeta_N^{(k)}(s) = 0. \end{cases} \tag{3.20}$$

Hence

$$\sum_{n=1}^{N-1} \frac{(\log n)^r}{n^s} = -\frac{(\log N)^r}{N^s}, \qquad (0 \leq r \leq k) \tag{3.21}$$

so that

$$\frac{(\log N)^r}{N^\sigma} \leq \sum_{n=1}^{N-1} \frac{(\log n)^r}{n^\sigma}, \qquad (0 \leq r \leq k) \tag{3.22}$$

or

$$1 \leq \sum_{n=1}^{N-1} \left(\frac{\log n}{\log N}\right)^r \left(\frac{N}{n}\right)^\sigma, \qquad (0 \leq r \leq k). \tag{3.23}$$

Since s is a zero of $\zeta_N(s)$, we certainly have that $\sigma < 2$. It follows that for any fixed N (3.23) can be true only if r is not too large. More explicitly, we must have, for example,

$$1 \leq \sum_{n=1}^{N-1} \left(\frac{\log(N-1)}{\log N}\right)^r \left(\frac{N}{n}\right)^2 = \tag{3.24}$$

$$= N^2 \left(\frac{\log(N-1)}{\log N}\right)^r \sum_{n=1}^{N-1} \frac{1}{n^2} <$$

$$< N^2 \frac{\pi^2}{6} \left(\frac{\log(N-1)}{\log N}\right)^r$$

from which it is easily seen that we may take

$$(3.25)\qquad \omega(N) = 1 + \frac{\log \frac{N^2\pi^2}{6}}{\log \frac{\log N}{\log(N-1)}}, \qquad (N\geq 3).$$

From (3.23) it is evident that one would like to have good upper bounds for the real parts of the zeros of $\zeta_N(s)$.
Defining

$$(3.26)\qquad \rho_N = \sup\left\{\sigma \in \mathbb{R} \mid \zeta_N(\sigma+it) = 0 \text{ for some } t \in \mathbb{R}\right\}$$

we have, c.f. SPIRA [6], $\rho_N < 1.85$.
For small N this upper bound may be improved as follows.
Defining

$$(3.27)\qquad \sigma_N = \sup\left\{\sigma \in \mathbb{R} \mid R_N(\sigma,t) = 0 \text{ for some } t \in \mathbb{R}\right\}$$

we clearly have

$$(3.28)\qquad \rho_N \leq \sigma_N .$$

For example, let us take N = 3. Then

$$(3.29)\qquad R_3(\sigma,t) = 1 + \frac{1}{2^\sigma}\cos(t \log 2) + \frac{1}{3^\sigma}\cos(t \log 3)$$

$$\geq 1 - \frac{1}{2^\sigma} - \frac{1}{3^\sigma} \overset{\text{def}}{=} f_3(\sigma).$$

Since $f_3(\sigma)$ is increasing and

$$(3.30)\qquad f_3(\sigma) = 0 \quad \text{for } \sigma = .787885....$$

it follows that

$$(3.31)\qquad \rho_3 < .7879.$$

Substitution of this estimate in (3.23) yields

$$(3.32)\qquad 1 \leq \sum_{n=1}^{2} \left(\frac{\log n}{\log 3}\right)^r \left(\frac{3}{n}\right)^{.7879} .$$

Since this inequality does *not* hold for r = 1 we have an alternative proof of the fact that all zeros of $\zeta_3(s)$ are simple.
Now let us consider the case N = 4. For N = 4 we have

(3.33) $$R_4(\sigma,t) = 1 + \frac{1}{2^\sigma}\cos(t\log 2) + \frac{1}{3^\sigma}\cos(t\log 3) + \frac{1}{4^\sigma}\cos(2t\log 2)$$

$$\geq 1 + \frac{1}{2^\sigma}\cos u - \frac{1}{3^\sigma} + \frac{1}{4^\sigma}\cos(2u) =$$

$$= 1 - \frac{1}{3^\sigma} + \frac{1}{2^\sigma}\cos u + \frac{1}{4^\sigma}(2\cos^2 u - 1) \geq$$

$$\geq 1 - \frac{1}{3^\sigma} - \frac{1}{4^\sigma} + \min_{|x|\leq 1}\left\{\frac{x}{2^\sigma} + \frac{2x^2}{4^\sigma}\right\} \geq$$

$$\geq 1 - \frac{1}{3^\sigma} - \frac{1}{4^\sigma} + \min_{v\in\mathbb{R}}(v + 2v^2) =$$

$$= \frac{7}{8} - \frac{1}{3^\sigma} - \frac{1}{4^\sigma} = f_4(\sigma).$$

Since $f_4(\sigma)$ is increasing and

(3.34) $$f_4(\sigma) = 0 \text{ for } \sigma = .669081\ldots$$

we find that

(3.35) $$\rho_4 < .6691.$$

Sustitution of (3.35) in (3.23) yields that all zeros of $\zeta_4(s)$ have multiplicity ≤ 2.
Similarly one may show that we may take

(3.36) $$\omega(5) = 5.$$

<u>CONJECTURE</u>. For every $N \geq 2$ all zeros of $\zeta_N(s)$ are simple.

In SPIRA [6] it was shown that $\rho_N < 1.85$. This estimate may be improved as follows.
If $\zeta_N(s) = 0$ then

(3.37) $$0 = |\zeta_N(s)| \geq 1 - \sum_{n=2}^{N} n^{-\sigma} > 2 - \zeta(\sigma)$$

so that

$$(3.38) \qquad \zeta(\sigma) > 2.$$

Since $\zeta(s)$ is decreasing on $s > 1$ we must have

$$(3.39) \qquad \sigma < p_0$$

where

$$(3.40) \qquad p_0 > 1 \text{ and } \zeta(p_0) = 2$$

One may verify that $p_0 = 1.728647....$

4. In this section we will show that for any costant $c_4 > 0$, $\zeta_N(s)$ does *not* have a zero in the triangle

$$(4.1) \qquad \begin{cases} 1 \le \sigma \le 1 + \dfrac{c_4}{\log^3 N} \\ \dfrac{2\pi}{\log N} \le t \le \dfrac{2\pi}{\log N} + \dfrac{\sigma-1}{\log^2 N} \end{cases}$$

if N is large enough. Compare SPIRA [5, p. 171]. In order to prove this we consider

$$R_N(t) = \sum_{n=1}^{N} \frac{1}{n} \cos(t \log n)$$

at the point $t = \dfrac{a\pi}{\log N}$, $(2 \le a \le 3)$.

By means of the Euler-Maclaurin summation formula we have

$$(4.2) \qquad R_N(t) = \int_1^N \frac{1}{x} \cos(t \log x)\, dx - \int_{1-}^{N+} \frac{1}{x} \cos(t \log x)\, d\,\varphi_1(x)$$

$$= I_1 + I_2.$$

The substitution $x = N^u$ in I_1 yields

$$(4.3) \qquad I_1 = \log N \int_0^1 \cos(t\,u \log N)\, du =$$

$$= \log N \frac{\sin(t \log N)}{t \log N} = \frac{\sin a\pi}{a\pi} \log N,$$

so that in view of $2 \le a \le 3$ we have

(4.4) $$I_1 \ge 0.$$

Furthermore we have

(4.5) $$I_2 = -\frac{\varphi_1(x)}{x}\cos(t\log x)\Big|_{1-}^{N+} + \int_1^N \varphi_1(x)\, d\,\frac{\cos(t\log x)}{x}$$

$$= \frac{1}{2} + \frac{1}{2N}\cos(a\pi) - \int_1^N \varphi_1(x)\,\frac{t\sin(t\log x)+\cos(t\log x)}{x^2}\,dx.$$

Now observe that

(4.6) $$\left|\int_1^N \varphi_1(x)\,\frac{t\sin(t\log x)}{x^2}\,dx\right| \le t^2\int_1^N \frac{1}{2}\,\frac{\log x}{x^2}\,dx \le$$

$$\le \frac{1}{2}\left(\frac{3\pi}{\log N}\right)^2\int_1^\infty \frac{\log x}{x^2}\,dx \to 0 \text{ as } N\to\infty,$$

and that, uniformly in t,

(4.7) $$\int_1^N \varphi_1(x)\,\frac{\cos(t\log x)}{x^2}\,dx \to \int_1^\infty \frac{\varphi_1(x)}{x^2}\,dx, \quad (N\to\infty).$$

Hence

(4.8) $$I_2 \to \frac{1}{2} - \int_1^\infty \frac{\varphi_1(x)}{x^2}\,dx, \quad (N\to\infty)$$

uniformly in t. Observing that

(4.9) $$\frac{1}{2} - \int_1^\infty \frac{\varphi_1(x)}{x^2}\,dx = \gamma$$

where $\gamma = .577415...$ is Euler's constant, it follows that

$$R_N(t) > \frac{1}{2}, \qquad \left(\frac{2\pi}{\log N} \le t \le \frac{3\pi}{\log N}\right) \tag{4.10}$$

if N is large enough. Next we observe that

$$\frac{\partial R_N(\sigma,t)}{\partial\sigma} = -\sum_{n=2}^{N} \frac{\log n}{n^{\sigma}} \cos(t \log n) \tag{4.11}$$

so that

$$\left| \frac{\partial R_N(\sigma,t)}{\partial\sigma} \right| \le \sum_{n=2}^{N} \frac{\log n}{n} \le c_5 \log^2 N, \qquad (\sigma\ge 1). \tag{4.12}$$

By the maximal slope principle we thus obtain from (4.10) and (4.12) that

$$R_N(\sigma,t) > 0 \tag{4.13}$$

on the rectangle

$$\begin{cases} 1 \le \sigma \le 1 + \dfrac{1}{4 \log^2 N} \\ \\ -\dfrac{2\pi}{\log N} \le t \le \dfrac{3\pi}{\log N} . \end{cases} \tag{4.14}$$

If N is large enough the triangle described by (4.1) is entirely contained in the rectangle described by (4.14), proving our assertion.

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